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Stochastic Interpretations of Nonrelativistic Quantum Theory[1]

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A critique of the causal and classical stochastic interpretations of nonrelativistic quantum mechanics is presented. The only way that the classical stochastic formulation can be made compatible with the theory of quantum measurement is to extend the probability measure density for fluctuating paths to the complex domain. In doing so, we obtain the generalized stochastic formulation in which the methods of classical probability theory can be used to describe the quantum mechanical phenomenon of interfering alternatives. Illustrative examples from quantum theory are used to show the complete compatibility between the traditional and generalized stochastic interpretations of quantum mechanics.

1. THE ORIGINS OF THE STOCHASTIC FORMULATIONS

Causal formulations of quantum theory grew out of the unwillingness to accept the fact that physical phenomena are not governed by physical laws but that only predictions of the expected values of each of various possible outcomes of the same experiment can be made. Yet, it is rather ironical that the causal interpretations found it necessary to rely upon a statistical interpretation of their results.

The aim of the causal formulations was to attribute a physical objectivity to the wave function which is the solution to the Schrödinger equation (Freistadt, 1957). Through the "delinearization" of the Schrödinger equation, two field equations for real velocity potentials in configuration space were obtained, to which a causal interpretation was given in the familiar

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language of hydrodynamics (Madelung, 1926; Takabayasi, 1952, 1953; Bohm and Vigier, 1954; Schöberg, 1954). It was not surprising that one of the field equations turned out to be the continuity equation since the basis of the causal interpretations consisted of the Schrödinger equation and the Born statistical interpretation of the wave function. Hence, by their very starting point, the causal formulations were circumvented by a statistical justification (Bohm, 1953). The other field equation had the formal appearance of a classical Hamilton–Jacobi equation and attempts were made to give a classical description to the motion even though the potential, appearing in the equation, was definitely nonclassical (Takabayasi, 1953, 1954; Bohm, 1952a). To this so-called "quantum" potential was attributed the origins of the nonclassical behavior of matter (i.e., the wave like properties and the existence of discrete stationary states).

Notwithstanding the conceptual appeal of the causal formulations, they posed more problems than they resolved (Freistadt, 1957). Rather than resulting from a limiting process, in the way that classical mechanics emerges from quantum mechanics in the limit of very short wavelengths, the causal formulations were a blend of classical and quantum concepts. The necessity of having to invoke two fields to describe the evolution of quantum systems which were inherently coupled to one another through the quantum potential is completely foreign to classical mechanics. In fact, it had to be considered as a new law of motion which had no analog in classical mechanics. The coupling afforded between the continuity equation and the Hamilton–Jacobi equation via the quantum potential elicited a "mysterious dependence of the individual on the statistical ensemble of which it is a member" (Freistadt, 1957).

Being causal, these formulations could not negate the possibility of measuring simultaneously both the position and velocity of the particle with an unlimited precision. Another apparent conclusion was that any arbitrary distribution, since it evolves according to the continuity equation, will remain invariant for all times, and, in general, it will not coincide with the so-called "equilibrium" distribution, dictated by the Born statistical interpretation of the wave function. According to the causal interpretations, the density of particles ρ was taken to be proportional to the density of the background fluid. And if the equilibrium distribution, $\rho = |\psi|^2$, where ψ is the wave function, did not hold initially, then it would be brought about in time through the presence of fluctuations (Bohm and Vigier, 1954). This appeared to be the only logical explanation of how the equilibrium distribution could be brought about in time. And "quantum fluctuations like classical fluctuations (e.g., Brownian motion) have basically the same origin —the chaotically complicated nature of the motion at the microscopic level" (Bohm, 1953).

It thus appeared that instead of having to treat an ensemble of particles, it might perhaps be possible to develop a completely classical formulation of quantum mechanics based upon the irregular motion of a single Brownian particle immersed in a suspension of lighter particles. This was later to develop into the classical stochastic formulation of nonrelativistic quantum mechanics. However, prior to a discussion of the classical stochastic formulation, several comments are in order regarding the causal description.

The common aspect of the proposed causal formulations was the decomposition of the complex velocity field into two real fields whose time evolution was governed by nonviscous Euler equations of hydrodynamics (de Broglie, 1927; Rosen, 1945; Bohm, 1952b). The transition from classical to quantum mechanics then lay in the "linearization" of the field equations. yet a prior knowledge of the Schrödinger equation was necessary in order to form the complex velocity field from the real components which satisfied the nonviscous Euler equations. Moreover, it is by no means apparent that the description afforded by two real velocity fields will be the same as a single, complex velocity field. In fact, we shall see that the two descriptions are, in general, not equivalent and, at most, the causal description can only achieve a partial success when one of the components of the complex velocity field vanishes. This attributes a preferential role to one of the velocity fields and it "behaves much more like an attached field than like an external field" (Freistadt, 1957). This conclusion can only be arrived at by considering the Schrödinger equation and not the real field equations themselves.

Another perplexing feature of the causal interpretation was the introduction of the classical concept of force into quantum mechanics. For then, every action there would necessarily be a reaction and it was the "reaction" which was completely foreign to quantum mechanics.

The classical concept of Brownian motion in nonrelativistic quantum mechanics was developed by Nelson (1966, 1967) and we shall refer to this formulation as the "classical" stochastic formulation. Nelson attempted to establish an equivalence between classical Markov processes and nonrelativistic quantum mechanics by deriving the Schrödinger equation from a Brownian motion process. According to the classical stochastic formulation, the kinematics of quantum processes would be described by the Einstein–Smoluchowski approximation of classical Brownian motion in configuration space while the dynamics would be governed by Newton's law of motion. We preface our remarks by noting that from a stochastic point of view, the two aspects of the theory are incompatible with one another. The stochastic process which is compatible with Newton's second law is not the Einstein–Smoluchowski process but rather the Ornstein–Uhlenbeck process which is a Markov process in phase space. Only in the limit of an infinitely large friction coefficient, in which the process tends to a limiting velocity, is the Einstein–Smoluchowski process a valid approximation to the Ornstein–Uhlenbeck process. Even if Nelson were to have considered a Brownian motion in the Einstein–Smoluchowski approximation he would have had to relegate the irreversibility of the motion. This Nelson did by considering the simultaneous existence of the reverse Brownian motion process; averaging both the forward and reverse Brownian motions he obtained a time-dependent condition for dynamical equilibrium which rendered the theory completely reversible (Lavenda, 1980). Although it is formally analogous to the Einstein condition of dynamic equilibrium in a suspension of colloidal particles in which the two irreversible effects of the drift and diffusion exactly balance one another, the density of particles is no longer time independent but rather satisfies the reversible continuity equation. The continuity equation resulted from an average of the forward and reverse Fokker–Planck equations.

In fact, the continuity equation and the condition for a time-dependent dynamical equilibrium are what actually constitute the basis of Nelson's theory and not the forward and reverse Markov processes considered individually. In order to account for the potential in Schrödinger's equation, Nelson had to introduce the concept of force, and this he did by postulating a time invariant form for the mean acceleration. In this manner, Nelson obtained the field equations of the causal theories, and by constructing the complex velocity field from these components, he derived the Schrödinger equation upon integration. Hence, the classical stochastic formulation is likewise valid only for bound states in which the unprivileged velocity component vanishes. And like the causal descriptions, the classical stochastic formulation fails to incorporate the phenomenon of interfering alternatives which gives rise to the wavelike properties of matter.

The classical stochastic formulation avoids any mention of the transition probability density, which along with the entire hierarchy of probability densities also satisfies the Fokker–Planck equation. For then it must be shown that, in the asymptotic time limit, the transition probability density transforms into the equilibrium or invariant distribution given by the Born probability relation. This would bring us back to the Bohm mechanism (Bohm and Vigier, 1954), whereby fluctuations are introduced to explain why any arbitrary initial distribution should decay to the equilibrium distribution in time.

It is now apparent that the classical stochastic formulation, which invokes a supposedly classical Markov process, is one of appearance rather than substance. As a result of the time-dependent condition of dynamical equilibrium, the probability density does not satisfy either of the Fokker–Planck equations separately but rather satisfies a time reversible continuity equation. This excludes the possibility of introducing a physical mechanism whereby any arbitrary initial distribution would decay to the equilibrium distribution, dictated by the Born statistical relation, merely with the passage of time.

Even if a unique correspondence could have been made between the evolution of two real velocity fields and the Schrödinger equation, there is more to quantum mechanics than the Schrödinger equation. One would still have to explain why certain quantities are not physical observables and why noncommuting physical observables are not simultaneously measurable with unlimited precision. It could be argued that quantum mechanics is only an approximation to a more fundamental theory, but this would only bring us back to hidden variable theories which have no physical confirmation.

As an alternative, we accept the view that there is a fundamental limit to physical measurements which is caused by the underlying "noise" present in the medium. Like the advocates of the statistical justification of the causal description, we can only say that it behaves in very much the same way as thermal fluctuations while emphasizing the completely different origins of the two phenomena. If we (i) realize that the inherent shortcoming of the causal formulations lay in the construction of real velocity fields, (ii) relinquish the requirement that the probability of a path be a real quantity, and (iii) recognize the fact that the velocity field in the stochastic analysis is an average quantity, then we are led to consider a complex measured, real diffusion process (Santamato and Lavenda, 1981). And *although the complex probability measure of a set of real paths will certainly not turn out to be the probability that a path belongs to the set, everything in nonrelativistic quantum theory works as if it were.*

The complex probability measure density for real path trajectories is responsible for the autoselectiveness of our generalized stochastic approach in that only physical observables turn out to be real quantities. On account of the inadequacies of the classical stochastic formulation, we are led to abandon the search for a purely classical interpretation of quantum theory with the consolation that by extending the probability measure to the complex domain, we are able to use the classical probability theory in a nonclassical context. Moreover, we shall see that the only conceptual difference between classical probability theory and the generalized stochastic approach lies in the fact that the drift vector is not a preassigned quantity but rather depends on the wave function. And it is through this complex velocity field that the "pilot" wave is able to guide the motion of the particle (de Broglie, 1927).

2. COMPLEX PROBABILITY MEASURES

The same problems that beset the Feynman path integral are also present in the definition of complex probability measures. This is not to say that the generalized stochastic approach is equivalent to the Feynman path integral formulation of nonrelativistic quantum mechanics. The fundamental difference is that whereas the path integral formulation deals with the probability amplitudes for possible events (Feynman and Hibbs, 1965), the generalized stochastic approach treats the stochastic processes giving rise to those events. In place of probability amplitudes, there are probability densities which are complex for those events which are not observable.

The formal similarity between the Feynman integral and the Wiener integral has long been known. If the mass m in the action of the Feynman integral is replaced by im, the (measure-theoretic) Wiener integral results (Nelson, 1964). The analyticity in m is established, and continues from positive-imaginary to positive-real values. This method is limited to only those cases which present extensive analyticity (Tarski, 1975). The problem can be stated in the following way. Denote $\mu_W(dx)$ as the Wiener measure and $\int_0^a \mu_W(dx)$ is the error integral. Alternatively if $\mu_F(dx) = \exp(-ix^2/2\sigma^2 t)/(-2i\pi\sigma^2 t)^{1/2}\,dx$ then $\int_0^a \mu_F(dx)$ is a linear combination of Fresnel integrals. The Wiener integral of a functional $f(x)$ is related formally to the Feynman integral by

$$\int \mu_W(dx) f\big((-ix)^{1/2}\big) = \int \mu_F(dx) f(x) \tag{1}$$

The problem is to give a rigorous definition to (1) so that $\mu_F(dx)$ can be interpreted as a complex probability measure. Expression (1) can be rewritten in the form

$$\int \mathscr{D}(x) e^{i\langle x, x\rangle/2} f(x)$$

where $\mathscr{D}(\cdot)$ is an infinite product of Lebesgue measures. The above form presupposes a Hilbert space. Itô (1966) considered the path integral in this form and showed asymptotic convergence (in the limit as the covariance operator tends to infinity) for a rather restricted class of functionals f with respect to potentials appearing in the Schrödinger equation.

Other definitions have appeared in the literature (cf. Tarski, 1975), each with its own limited range of validity, and it is not our purpose to select any one of these definitions as the definition of the generalized measure. Rather we take the pragmatic view that (1) exists and show that it gives the correct physics in every case.

The same formal definitions that apply to real probability measured stochastic processes apply equally as well to stochastic processes with complex probability measures. Let $\{X(\tau): \tau \in [s, t]\}$ denote a real stochastic process on the state space E^d $(d \geqslant 1)$. The state space is endowed with the σ algebra of Borel sets generated by the cylinder events

$$\{a_1 < x(\tau_1) < b_1 \cdots a_n < x(\tau_n) < b_n\}$$

for $s \leqslant \tau_1 < \cdots < \tau_n \leqslant t$. Suppose that a complex probability measure $\mu(dx)$ is induced on this σ algebra. Then the complex probability measure that a system starting at (y, s) is between a_1 and b_1 at time $\tau_1, \ldots,$ and is between a_n and b_n at time τ_n is

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_n, \tau_n | x_{n-1}, \tau_{n-1}) \cdots p(x_1, \tau_1 | y, s)\, dx_1 \cdots dx_n \tag{2}$$

where the complex transition probabilities p are the weight factors

$$p(x_{i+1}, \tau_{i+1} | x_i, \tau_i) = \frac{\exp[-i(x_{i+1} - x_i)^+ D^{-1}(x_{i+1} - x_i)/4(\tau_{i+1} - \tau_i)]}{\{|D|[-4i\pi(\tau_{i+1} - \tau_i)]^d\}^{1/2}} \tag{3}$$

in the complex measure (2) where $2D$ is the positive definite covariance matrix, $|\cdot|$ denotes the determinant and $^+$ the transpose.

In the generalized stochastic approach, the real standard Wiener process $W(t)$ is a complex measured process. It will henceforth be referred to as a complex measured, standard Wiener process which possesses all the formal statistical attributes of its real measured analog (i.e., stationarity and independent increments) with zero mean and

$$E\{W(t)W^+(s)\} = -iI(t-s) \tag{4}$$

where I is the unit matrix. This is to say that although $W(t)$ is a real Brownian motion, it has a complex probability associated with it.

The conditional probability that $W(t) \in \mathscr{B}$, a Borel set, at time t if it had the value $W(s) = y$ at the earlier instant s, is

$$\mathscr{P}\{W(t) \in \mathscr{B} | W(s) = y\} = \frac{1}{\{-4i\pi(t-s)\}^{d/2} |D|^{1/2}} \times \int_{\mathscr{B}} \exp\left[-\frac{i(x-y)^+ D^{-1}(x-y)}{4i(t-s)}\right] dx \tag{5}$$

If (5) had turned out to have been real, it would have contradicted quantum theory, which asserts that only one measurement can be made on the *same* system. In this sense, a complex probability measure for real diffusion process stands in the defense of the quantum theory of measurement. The complex probability (5) is known to coincide with the complex transition probability

$$P(\mathscr{B}, t \mid y, s) = \mathscr{P}\{W(t) \in \mathscr{B} \mid W(s) = y\} \tag{6}$$

and since the process is Markov, it satisfies the Chapman–Kolmogorov relation

$$P(\mathscr{B}, t \mid y, s) = \int_{E^d} P(\mathscr{B}, t \mid z, \tau) P(z, \tau \mid y, s)\, dz \tag{7}$$

Provided the density

$$p(x, t \mid y, s) = P(dx, t \mid y, s)/dx \tag{8}$$

exists, it too will satisfy the Chapman–Kolmogorov relation.

The Chapman–Kolmogorov relation for the density (8) is formally analogous to the propagator relation in the Feynman path integral formulation. However, whereas the kernel is the propagator of the wave function [cf. equation (38)] or the probability amplitude, the complex transition probability density (8) is the propagator of the real probability density (Santamato and Lavenda, 1981)

$$\rho(x, t) = \int_{E^d} p(x, t \mid y, s) \rho(y, s)\, dy \tag{9}$$

It is a remarkable feature of the generalized stochastic formulation that (9) is always a real quantity even though the transition probability density is necessarily complex.

3. QUANTUM MECHANICAL DIFFUSION PROCESSES

In this section we derive an explicit formula for the complex transition probability density for a given quantum mechanical diffusion process. Let $X(\tau)$ be a real diffusion process on the interval $[s, t]$ with values on E^d. The decisive properties of diffusion processes which are Markov with almost certainly continuous sample functions $X(\omega, \cdot)$ (where ω is the sample tag which will usually be suppressed), is that their transition probability is uniquely determined by the first two moments of the distribution; that is, the drift vector

$$E_{y,s}\{X(t)-X(s)\}=b(y,s)(t-s)+O(t-s) \qquad (10)$$

and the diffusion matrix D (Arnold, 1974), where

$$E_{y,s}\{[X(t)-X(s)][X(t)-X(s)]^{+}\}=-2iD(t-s)+O(t-s) \qquad (11)$$

where $E_{y,s}$ denotes the conditional expectation with respect to the complex probability measure under the condition that $X(s)=y$ as $t\downarrow s$. Hence, b is the mean velocity vector of the random motion described by the real diffusion process $X(t)$ and D is a measure of the local magnitude of the fluctuation of $X(t)-X(s)$ about the mean value.

The diffusion process is governed by the stochastic differential equation[2]

$$dX(t)=b(X(t),t)\,dt+(2D)^{1/2}\,dW(t), \qquad X(s)=y \qquad (12)$$

where $W(t)$ is the complex measured Wiener process introduced in Section 2. Classically speaking, equation (12) would correspond to the Einstein–Smoluchowski process which is derived from the Ornstein–Uhlenbeck process in the limit of an infinitely large friction coefficient. No such approximation exists in the generalized stochastic formulation. The Ornstein–Uhlenbeck process describes the phase space Brownian motion in an external field of force. In quantum mechanics, forces do not act on particles to cause a change in their motion. The fundamental classical notion of regarding forces as the cause of changes in the motion is replaced in quantum mechanics by momentum as being the direct cause in conjunction with the de Broglie relations. In the generalized stochastic formulation, the potentials of the field of force are seen to modify the probability of finding the particle in any given state through the complex probability measure for the set of paths. This is entirely consistent with quantum theory, which negates the possibility of applying Newton's law of motion to a single electron through the assertion that momentum and position cannot exist under conditions in which they are simultaneously measurable with unlimited precision.

Quantum theory places a lower bound on the limit of the precision of this measurement and it is natural to take Planck's constant $\hbar$ (divided by 2π) as a measure of the strength of the random fluctuations by setting it to be proportional to the diffusion coefficient. Furthermore, in order to obtain the correct limiting behavior in the case of large mass where the motion

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[2] This equation is to be interpreted in a purely formal sense since the drift is an already averaged quantity with respect to a complex probability distribution (cf. Eq. (10)).

should become classical, the diffusion coefficient should also be inversely proportional to the mass, viz.,

$$2D = \frac{\hbar}{m} I \tag{13}$$

which has long been known to be the connection between Brownian motion and quantum mechanics (Fürth, 1933).

In summary, equation (12) is not to be considered as an approximation to a more fundamental description of the motion in the way that the Einstein–Smoluchowski process is an approximation to the Ornstein–Uhlenbeck process. Rather, it is the momentum which controls statistically the relation between events at different times, according to the stochastic differential equation (12) (Lavenda and Santamato, 1980a,b).

The complex transition probability can now be derived by employing the property of the absolutely continuous substitution of (complex) probability measures for diffusion processes with the same local variances but with different drifts. To this end, we compare the quantum mechanical diffusion process (12) with the free diffusion process

$$d\tilde{X}(t) = (2D)^{1/2} dW(t), \qquad \tilde{X}(s) = y \tag{14}$$

The process $\tilde{X}(t)$ has the same sample functions as the process $X(t)$ but it is considered as a distinctly different stochastic process with respect to its complex probability measure $\tilde{\mu}(dx)$, as opposed to the complex probability measure $\mu(dx)$ of the quantum diffusion process $X(t)$.

The transformation of the free diffusion process $\tilde{X}(t)$ into the quantum diffusion process $X(t)$ provides us with a nice interpretation of the quantum theory of measurement when it is viewed as arising from the interaction of the system with the measuring apparatus. Prior to the interaction, quantum theory tells us that the system does not have a well-defined value but rather it possesses a latent potentiality of assuming any one of a set of possible values. Through the process of interaction between the system and the measuring apparatus or external field, the process described by (14) is converted into a new process described by (12). The average momentum imparted to the system as it passes through a region in which there is a scalar potential $\mathscr{S}$ and vector potential $\mathscr{A}$ is

$$mb = \nabla \mathscr{S} + \mathscr{A} \tag{15}$$

The presence of the external fields serve to modify the relative likelihood of the paths of the particle and consequently the probability of observing the system in any given state.

Since μ is absolutely continuous with respect to $\tilde{\mu}$, the probability measure density

$$\frac{d\mu}{d\tilde{\mu}}[\tilde{X}(\omega,\cdot)] = \rho(\tilde{X}(\omega,\cdot)) \tag{16}$$

exists where $\tilde{X}(\omega,\cdot)$ is an E^d-valued function on $[s,t]$ (i.e., a sample function). The Radon–Nikodym derivative (16) is given by the Girsanov formula (19) (Girsanov, 1960), which can be rationalized in the following manner. The problem is one of deciding between two (complex) probability measures μ and $\tilde{\mu}$ on the basis of "observing" a sample function $X(\omega,\tau)$, $s \leqslant \tau \leqslant t$. The increment in the likelihood ratio (16) due to the increment in the process $d\tilde{X}$ is

$$d\rho = i\rho mb\, d\tilde{X}/\hbar \tag{17}$$

which can be viewed to be analogous to the Feynman postulate relating the probability amplitude to the action. On account of the statistical correlations between the system and the measuring apparatus, represented by the presence of the external fields, $d\rho/\rho \neq d\ln\rho$ but rather using the Itô differentiation rule, we have

$$d\ln\rho = d\rho/\rho - \frac{1}{2}\left(\frac{d\rho}{\rho}\right)^2 = \frac{im}{\hbar}\left(b^+ d\tilde{X} - \frac{m}{\hbar} b^+ D b\, d\tau\right) \tag{18}$$

Then integrating on the interval $[s,t]$ we obtain the Girsanov formula:

$$\rho[\tilde{X}(\omega,\tau), s \leqslant \tau \leqslant t] = \exp\left\{\frac{im}{\hbar}\int_s^t \left(b^+ d\tilde{X} - \frac{1}{2} b^+ b\, d\tau\right)\right\} \tag{19}$$

Due to the presence of the Itô stochastic integral, expression (19) for the complex probability measure density is in a somewhat inconvenient form. Use of the Itô formula

$$\int_s^t \nabla \mathscr{S}^+\, d\tilde{X} = \mathscr{S}(\tilde{X}(t)) - \mathscr{S}(\tilde{X}(s)) - \int_s^t (\partial_\tau \mathscr{S} - iD : \nabla\nabla \mathscr{S})\, d\tau \tag{20}$$

and the definition of the Fisk–Stratonovich symmetric integral

$$\int_s^t \mathscr{A}^+ \circ d\tilde{X} = \int_s^t \mathscr{A}^+\, d\tilde{X} - i\int_s^t D : \nabla \mathscr{A}\, d\tau \tag{21}$$

allows us to cast expression (19) in the form (Lavenda, 1982)

$$\rho[\tilde{X}(\omega,\tau), s \leqslant \tau \leqslant t] = \exp\left\{\frac{i}{\hbar}\left[\mathscr{S}(\tilde{X}(t)) - \mathscr{S}(\tilde{X}(s)) + \int_s^t \mathscr{A}^+ \circ d\tilde{X} + V(\tilde{X}(\tau))\, d\tau\right]\right\} \tag{22}$$

where we have set

$$\partial_\tau \mathscr{S} + \hbar^{-1} D : (\nabla \mathscr{S} + \mathscr{A})(\nabla \mathscr{S} + \mathscr{A})^+ - iD : \nabla(\nabla \mathscr{S} + \mathscr{A}) = -V(x) \tag{23}$$

The connection with quantum mechanics can now be achieved by linearizing equation (23) through the transformation

$$\mathscr{S} = -i\hbar \ln \psi \tag{24}$$

to obtain the Schrödinger equation

$$i\hbar \partial_\tau \psi = -\frac{1}{2m}(-i\hbar\nabla + \mathscr{A})(-i\hbar\nabla + \mathscr{A})\psi + V(x)\psi \tag{25}$$

The Schrödinger equation (25) identifies V as the scalar potential appearing in the generalized Hamilton–Jacobi equation (23) [cf. discussion following Equation (91)].

Expression (22) is the most general expression for the complex probability measure density for the transformation of the free diffusion process $\tilde{X}(t)$ into the quantum mechanical diffusion process $X(t)$. It should be borne in mind that the functional (22) is determined uniquely in terms of the scalar potential V and the vector potential $\mathscr{A}$. In contrast to classical mechanics where the external fields act through the force to create a change in the motion of the particle, in quantum mechanics they alter the probability measure density of the path trajectories of the particle. This fundamental quantum mechanical property of the probabilistic effect of external fields is made apparent by expressing the complex transition probability density in terms of a conditional expectation of the complex probability measure density (22).

The (complex) probability that the solution $\tilde{X}_{ys}(t)$ of the free diffusion process (14) is found in the Borel set $\mathscr{B}$ at time t, if it was in the state y at the earlier instant s, coincides with the function

$$\mathscr{P}\{\tilde{X}_{ys}(t) \in \mathscr{B}\} = \tilde{P}(\mathscr{B}, t \mid y, s) \tag{26}$$

By virtue of the Radon–Nikodym theorem, the transition probabilities P and $\tilde{P}$, for the quantum and free diffusion processes, are related by (Gihman and Skorohod, 1972)

$$P(\mathscr{B}, t|y, s) = \int \chi_{\mathscr{B}}(x) \rho(x, t|y, s) \mathscr{P}\{\tilde{X}_{ys}(t) \in dx\} \tag{27}$$

where χ is the indicator and $\rho(x, t|y, s)$ is the conditional expectation

$$\rho(x, t|y, s) = \exp\left\{\frac{i}{\hbar}[\mathscr{S}(x) - \mathscr{S}(y)]\right\} \cdot E\left\{\exp\left[\frac{i}{\hbar}\int_s^t (\mathscr{A}^+ \circ d\tilde{X} + V\, d\tau)\right] \middle| \tilde{X}_{ys}(t) = x\right\} \tag{28}$$

Since the complex transition probability for the free diffusion process is given by (5), viz.,

$$\tilde{P}(\mathscr{B}, t|y, s) = \frac{\{-4i\pi(t-s)\}^{-d/2}}{|D|^{1/2}} \int_{\mathscr{B}} \exp\left[-\frac{i(x-y)^+ D^{-1}(x-y)}{4(t-s)}\right] dx \tag{29}$$

expression (27) can be written as

$$P(\mathscr{B}, t|y, s) = \frac{\{-4i\pi(t-s)\}^{-d/2}}{|D|^{1/2}} \times \int_{\mathscr{B}} \rho(x, t|y, s) \exp\left[-\frac{i(x-y)^+ D^{-1}(x-y)}{4(t-s)}\right] dx \tag{30}$$

or in terms of its density we have [cf. equation (8)]

$$p(x, t|y, s) = \rho(x, t|y, s)\tilde{p}(x, t|y, s) \tag{31}$$

where $\tilde{p}$ is the complex transition probability density of the free diffusion process.

For given external fields, V and $\mathscr{A}$, the generalized Hamilton–Jacobi equation (23) can, in principle, be solved for the action $\mathscr{S}$. In many cases, it is easier to solve the linear equation (25) and obtain $\mathscr{S}$ through the logarithmic transformation (24). Hence, the complex transition probability density of the quantum mechanical diffusion process is uniquely determined in terms of the applied external fields V and $\mathscr{A}$. And since the complex transition probability density is the propagator of the probability density according to (9), it is clearly seen that the external fields will influence the probability distributions rather than having the classical effect of acting directly on the particle to change its motions via the force they create.

The complex transition probability density expression (31) affords a direct connection with the Feynman propagator (Feynman and Hibbs, 1965). Introducing the wave function ψ according to (24) into the conditional expectation (28), the complex transition probability density (31) can be written in the form

$$p(x,t|y,s)=\frac{\psi(x,t)}{\psi(y,s)}K^*(x,t|y,s) \tag{32}$$

where K^* is the complex conjugate of the Feynman propagator

$$\begin{aligned} K^*(x,t|y,s) &= \{-2i\pi\hbar(t-s)/m\}^{-d/2} \\ &\quad \times E\left\{\exp\left[\frac{i}{\hbar}\int_s^t(\mathscr{A}^+\circ d\tilde{X}+V\,d\tau)\right]\middle|\tilde{X}_{ys}(t)=x\right\} \\ &\quad \times \exp\left[-\frac{im(x-y)^+(x-y)}{2\hbar(t-s)}\right] \end{aligned} \tag{33}$$

Expression (33) is easily recognized as the Feynman–Kac formula for the Green's function of the complex conjugate Schrödinger equation with the initial condition

$$\lim_{t\downarrow s}K^*(x,t|y,s)=\lim_{t\downarrow s}p(x,t|y,s)=\delta(x-y) \tag{34}$$

The complex transition probability density satisfies the pair of Kolmogorov equations

$$\partial_t p=-\nabla\cdot bp-\frac{i\hbar}{2m}\Delta p \tag{35}$$

and

$$-\partial_s p=b\cdot\nabla' p-\frac{i\hbar}{2m}\Delta' p \tag{36}$$

where $\Delta\equiv\nabla^2$ and the prime means differentiation with respect to the initial coordinates of the transition. The Fokker–Planck equation (35) is to be solved with respect to the initial condition (34) and the backward Kolmogorov equation (36) has the end point condition

$$\lim_{s \uparrow t} p(x, t | y, s) = \delta(x - y) \tag{37}$$

The derivation of the pair of Kolmogorov equations is based on the Chapman–Kolmogorov equation (7). In an analogous fashion, Feynman (1948) used the propagator expression

$$\psi(x, t) = \int K(x, t | y, s) \psi(y, s)\, dy \tag{38}$$

to show that ψ satisfies the Schrödinger equation. But whereas Feynman postulated the form of the kernel based on a suggestion by Dirac (1933) that "...it would seem desirable to take up the question of what corresponds in the quantum theory to the Lagrangian method of the classical theory," we have obtained an explicit expression for the propagator [cf. equation (33)] from the fact that we have introduced the quantum mechanical diffusion process (12) which is lacking in the Feynman path integral formulation.

Compatibility with traditional quantum mechanics can be demonstrated by introducing (32) into the complex Fokker–Planck equation (35). We then obtain

$$\partial_t p = -\nabla \cdot J \tag{39}$$

where J is the complex transition probability current density

$$J(x, t | y, s) = \frac{\hbar}{im} \frac{1}{\psi(y, s)} \{ K^* \nabla \psi(x, t) - \psi(x, t) \nabla K^* \} \tag{40}$$

Expanding the wave function in a complete set of eigenfunctions, $\phi_n(x)$,

$$\psi(x, t) = \sum_n c_n \phi_n(x) \exp(-iE_n t/\hbar) \tag{41}$$

with the corresponding eigenvalues E_n and expressing the complex conjugate of the kernel as a bilinear sum of eigenfunctions,

$$K^*(x, t | y, s) = \sum_m \phi_m^*(x) \phi_m(y) \exp[iE_m(t - s)/\hbar] \tag{42}$$

we can write (40) in the form

$$J(x,t|y,s)=\frac{\hbar}{im}\frac{1}{\psi(y,s)}\left\{\sum_{m,n}c_n e^{-iE_m s/\hbar}\phi_m(y)[\phi_m^*\nabla\phi_n-\phi_n\nabla\phi_m^*]\right\} \tag{43}$$

In the case where there is no interference of the contribution of different stationary states, expression (43) reduces to the well-known expression for the *real* probability current density. The evolution of quantum systems is therefore due to the nonclassical phenomenon of interfering alternatives. It also follows from (42) that the transition probability density is properly normalized viz.,

$$\int p(x,t|y,s)\,dx=\frac{1}{\psi(y,s)}\sum_n\phi_n(y)e^{-iE_n s/h}\int\phi_n^*(x)\phi_n(x)\,dx=1 \tag{44}$$

The nonclassical phenomenon of interfering alternatives can now be derived in a completely classical manner. Let p_α be the complex transition probability density for the system to make a transition to the state $\phi_\alpha(x,t)$ if it was known to be in the state $\psi(y,s)$ at the earlier time s. According to (32), this transition probability density is given by

$$p_\alpha(x,t|y,s)=\frac{\phi_\alpha(x,t)}{\psi(y,s)}K^*(x,t|y,s) \tag{45}$$

Similarly, for the transition to the state $\phi_\beta(x,t)$ we have

$$p_\beta(x,t|y,s)=\frac{\phi_\beta(x,t)}{\psi(y,s)}K^*(x,t|y,s) \tag{46}$$

Denote by c_α and c_β the complex probabilities for the events α and β. Note that the only difference with classical probability theory is that we cannot impose the condition $c_\alpha+c_\beta=1$ since the transition probability density is complex in quantum mechanics. Now the complex transition probability density for either event α or β is given by the formally classical expression

$$p(x,t|y,s)=c_\alpha p_\alpha(x,t|y,s)+c_\beta p_\beta(x,t|y,s) \tag{47}$$

In order to show that (47) is in complete agreement with the quantum mechanical interpretation of interfering alternatives where the probability density is given by

$$\rho(x,t)=|c_\alpha\phi_\sigma(x,t)+c_\beta\phi_\beta(x,t)|^2 \tag{48}$$

we multiply (47) by $\rho(y, s)$ and integrate over y:

$$\begin{aligned}\int p(x, t|y, s)\rho(y, s)\,dy &= \int (c_\alpha p_\alpha + c_\beta p_\beta)\rho(y, s)\,dy \\ &= \left[c_\alpha\phi_\alpha(x, t) + c_\beta\phi_\beta(x, t)\right] \\ &\quad \times \int K^*(x, t|y, s)\psi^*(y, s)\,dy \\ &= |c_\alpha\phi_\alpha + c_\beta\phi_\beta|^2 = \rho(x, t) \end{aligned} \tag{49}$$

In the derivation of (49) we have used expressions (38), (45), and (46). It is also evident from (49) that the complex transition probability density is the propagator of the probability density [cf. equation (9)] in the exact same way that the kernel is the propagator of the wave function [cf. equation (38)]. Furthermore, (49) shows how the quantum mechanical phenomenon of interfering alternatives can be accounted for by combining (complex) probability densities according to the methods of classical probability theory. In Section 5, we shall illustrate how quantum phenomena can be explained in this way.

Having developed the mathematical apparatus of quantum diffusion processes we are in a position to investigate the stochastic structure of quantum theory. However, prior to such an investigation, it will prove advantageous to compare the stochastic and causal formulations.

4. STOCHASTIC VERSUS CAUSAL INTERPRETATIONS OF QUANTUM THEORY

In many respects, the generalized stochastic formulation can be considered as an outgrowth of the statistical justification of the causal interpretation of quantum mechanics. A comparison between the two interpretations will serve to illustrate the inherent limitations of the causal theories to encompass the entire spectrum of quantum phenomena.

The classical statistical interpretations of quantum mechanics (Ghirardi et al., 1978) attempted to extract real field equations from the complex Schrödinger equation through the Ansatz:

$$\psi(x, t) = \exp\left\{\frac{i}{\hbar}[S(x, t) - iR(x, t)]\right\} \tag{50}$$

where S and R are real fields. In comparison with the generalized stochastic formulation, this means [cf. equation (24)]

$$\mathscr{S}(x,t)=S(x,t)-iR(x,t) \tag{51}$$

Consequently, the complex drift vector (15) is given by

$$b=\frac{1}{m}[\nabla S+\mathscr{A}-i\nabla R] \tag{52}$$

Observe that a component of the vector potential is absent in the imaginary component of the drift. This will be seen to be a consequence of the fact that all external fields must be real in order for the probability distribution to be real and a conserved quantity (cf. the discussion at the end of this section).

Introducing (51) into the generalized Hamilton–Jacobi equation (23) results in the pair of field equations

$$\partial_t R+\frac{2}{\hbar}D:\nabla R(\nabla S+\mathscr{A})^{+}+D:\nabla(\nabla S+\mathscr{A})=0 \tag{53}$$

$$\partial_t S+\frac{1}{\hbar}D:(\nabla S+\mathscr{A})(\nabla S+\mathscr{A})^{+}-D:\left[\frac{1}{\hbar}\nabla R\nabla R^{+}+\nabla\nabla R\right]=-V \tag{54}$$

Equations (53) and (54) are to be solved for the given initial conditions

$$S(x,0)=S_0(x) \quad \text{and} \quad R(x,0)=R_0(x) \tag{55}$$

That is, the initial conditions are given for the configurations space fields R and S instead of the initial conditions for a system of particles (Keller, 1953).

The physical interpretation of the field equation (53) is straightforward. Multiplying (50) by its complex conjugate and using the Born probability relation $\rho=|\psi|^2$, we obtain

$$R(x,t)=\tfrac{1}{2}\hbar\ln\rho(x,t) \tag{56}$$

Denoting

$$v=\frac{1}{m}(\nabla S+\mathscr{A}) \tag{57}$$

as the current velocity and inserting expressions (56) and (57) into the field equation (53) results in

$$\partial_t\rho+\nabla\cdot\rho v=0 \tag{58}$$

provided the diffusion matrix is given by (13).

The second field equation (54) has the form of a classical Hamilton–Jacobi equation with the additional scalar potential

$$U(x,t) = -\frac{1}{2m}\left[(\nabla R)^2 + \hbar\,\Delta R\right] = \frac{\hbar^2}{4m}\left[\tfrac{1}{2}(\nabla \ln\rho)^2 - \rho^{-1}\Delta\rho\right] \tag{59}$$

The time-dependent scalar potential (59) was first commented on by de Broglie (1927), who sought to establish a quantum description of matter as a modified form of classical mechanics. His interpretation developed into the so-called "pilot-wave" theory. It was rederived by Bohm (1952b) in his causal interpretation and has come to be known as the "quantum" or Bohm potential.

Especially in the presence of the vector potential $\mathscr{A}$, it is more instructive to consider the physical Euclidean space E^3 and identify $\mathscr{A}$ as $-(e/c)\mathbf{A}$, where $\mathbf{A}$ is the electromagnetic vector potential, e is the charge, and c is the speed of light. On taking the gradient of equation (54) and using the field equations

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A} - \nabla\Phi \tag{60}$$

where $\mathbf{B}$ is the magnetic field strength and Φ is the scalar potential, we obtain

$$m d_t\mathbf{v} = -\nabla V + e\nabla\Phi + \mathbf{F} - \nabla U \tag{61}$$

The substantial derivative has been denoted by d_t and $\mathbf{F}$ is the Lorentz force

$$\mathbf{F} = e\left[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})\right] \tag{62}$$

Therefore, the separation of the complex field (50), satisfying Schrödinger's equation (25), yields the continuity equation (58) and the Euler equation (61) for a nonviscous fluid under the influence of a scalar potential (59) which displays a somewhat mysterious dependence on the density and its gradient. For the particular case of a fluid at rest, $\mathbf{v} = 0$, stationary quantum states of the system result. Since $R = \hbar \ln\psi$ we obtain

$$\frac{\hbar^2}{2m}\Delta\psi - V\psi = \varepsilon\psi \tag{63}$$

upon integrating the Euler equation (61). The constant of integration, ε, has dimensions of energy. The fact that the stationary Schrödinger equation (63) results when $S=0$ led the adherents of the causal interpretation to the conclusion that R behaves much more like an attached field than S (Freistadt, 1957). This is to say that it is possible to conceive of an electron without an S field but not without an R field.

In the framework of the classical stochastic approach, it will be appreciated that the decoupling of the field equations by setting $S=0$ corresponds to quantum mechanical bound states when a time analytic continuation is performed on real Markov processes (Guerra and Ruggiero, 1973). When $S \neq 0$, it cannot account for the way quantum theory describes the change in probability in time due to the interference of the contributions of different stationary states. In summary, there is no way in which two real fields can interact so as to account for quantum phenomena which are described by a single complex field. It is precisely the complex nature of this field that thwarts a purely classical interpretation.

The pair of hydrodynamical equations, (53) and (54), were rederived in the classical stochastic formulation which sought to establish an equivalence between classical Markov processes and quantum mechanics (Nelson, 1966). In contrast to the causal interpretations which seek a decomposition of the Schrödinger equation for the complex ψ field into two real field equations, the classical stochastic formulation sought to derive the Schrödinger equation from a Brownian motion process on configuration space. Whereas in the causal interpretation, the statistical implications followed as a consequence of the continuity equation (58), the classical stochastic formulation introduced the statistics through the stochastic nature of the particle motion. According to the causal interpretation, the observable particle density represented the supposedly unobservable background fluid in which $\rho = |\psi|^2$ represented the equilibrium distribution of particles. If $\rho \neq |\psi|^2$ initially then it would be brought about in time through the action of random fluctuations. In fact, Bohm (1953) went to the extent of drawing an analogy between $|\psi|^2$ and an equilibrium Gibbs ensemble and invoked an H theorem for its establishment. However, such an irreversible mechanism is lacking in the time reversible field equations (58) and (61) and no such mechanism is compatible with the evolution they describe.

The classical stochastic approach attempted to establish an equivalence between real Markov diffusion processes and quantum mechanics (Nelson, 1966, 1967). The particle was assumed to undergo Brownian motion in configuration space, being described by the stochastic differential equation

$$dX(t) = \beta(X(t), t)\,dt + (2\nu)^{1/2}\,dW(t) \tag{64}$$

where $W(t)$ is a *real* probability measured (standard) Brownian motion and $dW(t)$ is independent of $X(\tau)$ for $\tau \leqslant t$. The real drift vector β is the mean forward derivative of the process $X(t)$, viz.,

$$\mathscr{D}X(t) \equiv \lim_{\tau \downarrow 0} \tau^{-1} E\{X(t+\tau) - X(t) | \mathsf{P}_t\} = \beta(x(t), t) \tag{65}$$

where the conditional expectation is taken with respect to an increasing family of σ algebra of events P_t, generated by the process up to time t. It is quite evident that the irreversible diffusion process (64) cannot be made compatible with the reversible behavior of quantum phenomena so that the classical stochastic formulation was constrained to consider the corresponding time-reversed diffusion process

$$dX(t) = \beta_*(X(t), t)\, dt + (2\nu)^{1/2}\, dW_*(t) \tag{66}$$

where W_* has the same statistical properties as W with the exception that $dW_*(\tau)$ is independent of $X(t)$ for $t \geqslant \tau$. The mean backward velocity is defined as

$$\mathscr{D}_* X(t) \equiv \lim_{\tau \downarrow 0} \tau^{-1} E\{X(t) - X(t-\tau) | \mathsf{F}_t\} = \beta_*(x(t), t) \tag{67}$$

where the conditional expectation is now taken with respect to a decreasing family of σ algebras F_t, generated by the process down to time t. In other terms, P_t represents past events while F_t represents future events.

It should now be appreciated that if the Brownian motion is to satisfy *simultaneously* both (64) and (66), it cannot satisfy either of these processes separately but rather an average of the two (Lavenda, 1980). This artifice is used to eliminate the irreversible elements contained in both the forward and reverse diffusion processes. The averaging is made on the corresponding Fokker–Planck equations which yields both the time reversible continuity equation (58) and a time-dependent condition for dynamical equilibrium [cf. equation (76)].

Let us consider instead the Fokker–Planck equations for the forward and reverse processes that are satisfied by the transition probability density. This will serve to shed light on the physical nature of the underlying assumptions. The real transition probability density p satisfies the forward Fokker–Planck equation

$$\partial_t p = -\nabla \cdot \beta p + \nu \Delta p \tag{68}$$

Following Doob (1953), we define the reverse transition probability density p_* by

$$p_*(y, s | x, t) \rho(x, t) = p(x, t | y, s) \rho(y, s) \tag{69}$$

Depending on how one specifies the probability density ρ, one of the two transition probability densities in (69) will be dependent on this choice. For if we take ρ as the initial, equilibrium distribution, then the transitional probability density p can be chosen quite independently of ρ. But once p is fixed, the reverse transition probability density p_* will, in general, be dependent on ρ. Alternatively, if we choose ρ such that

$$\lim_{|t-s|\to\infty} p(x,t|y,s)=\rho(x,t) \tag{70}$$

then a change in ρ will force a change in p. In this event, p_* can be chosen quite independently of ρ. It should also be borne in mind that in order to describe quantum phenomena, neither the forward process (64) nor the reverse process (66) is stationary in the wide sense; the drift vectors have an explicit dependence on time.

Introducing definition (69) into the forward Fokker–Planck equation (68) leads to

$$\partial_t p_* = -\beta_*\cdot\nabla p_* + \nu\Delta p_* \tag{71}$$

which is the time-reversed Fokker–Planck equation where

$$\beta_* = \beta - 2\nu\nabla\ln\rho \tag{72}$$

It is no longer possible to average the forward and reverse Fokker–Planck equations, as is done in the classical stochastic formulation for the probability density, since the forward and reverse transition probability densities are, in general, different. However, we can make appeal to the principle of dynamic reversibility (Tolman, 1938), which guarantees that for any possible forward motion there will be a reverse of that motion such that the same values of the coordinates are reached but in reverse order and with reversed values of the velocities. In order to apply the principle of dynamic reversibility, it is convenient to decompose the drift vector into components on the basis of their symmetry under time inversion, viz.,

$$\beta(x(t),t)=v(x(t),t)+u(x(t),t) \tag{73}$$

where $v\to -v$ and $u\to u$ under time reversal. Since for Markov processes, the past P_t is equal to the future for reverse times F_{-t} (Doob, 1953), it follows that

$$\begin{aligned}\mathscr{D}_* X(-t) &= \lim_{\tau\downarrow 0}\tau^{-1}E\{X(-t)-X(-t-\tau)|\mathsf{F}_{-t}\} = \beta_*(x(-t),-t)\\ &= \lim_{\tau\downarrow 0}\tau^{-1}E\{X(t)-X(t+\tau)|\mathsf{P}_t\} = -\beta(x(t),t) = -\mathscr{D}X(t)\end{aligned} \tag{74}$$

where we have used the property that the process $X(t)$ is invariant under time reversal. From the decomposition of the forward drift (73) and relation (74), we conclude that

$$\beta_*(x(t),t) = -u(x(t),t) + v(x(t),t) \tag{75}$$

and substituting (73) and (75) into (72) we obtain

$$u = \nu\nabla \ln \rho \tag{76}$$

Expression (76) is formally analogous to the Einstein condition of dynamical equilibrium. In the Einstein (1956) theory of Brownian motion, u is the fluctuating or osmotic velocity which is the velocity acquired by a Brownian particle due to the action of a (virtual) force that is needed to balance the osmotic force. The condition (76) is only formally analogous to the condition of dynamical equilibrium since the probability density is an explicit function of time in the classical stochastic interpretation. This condition has the effect of reducing the forward Fokker–Planck equation (68) to the reversible continuity equation (58) and attests to the fact that averaging the forward and reverse Brownian motion renders a new process which is completely reversible. In fact, the derivation of the Schrödinger equation is based on the reversible continuity equation (58), the formal condition of dynamic equilibrium (76), and a particular definition of the mean acceleration.

The hydrodynamic equations for the two velocity fields, u and v, now follow straightforwardly by differentiating (76) with respect to time and eliminating the time derivative of the density with the aid of the continuity equation (58). This results in

$$\partial_t u = -\nabla(u\cdot v) - \nu\nabla(\nabla\cdot v) \tag{77}$$

which is the gradient of the field equation (53) for a diffusion matrix $D = \nu I$. In order to derive the equation of motion for v, we need a definition of the acceleration. This serves to introduce the force $-\nabla V$ through Newton's law of motion. Nelson (1966, 1967) argued that the only definition of the mean acceleration which does not distinguish between the forward and reverse direction of time is the symmetric combination

$$\tfrac{1}{2}[\mathscr{D}\mathscr{D}_* X(t) + \mathscr{D}_*\mathscr{D}X(t)] = a \tag{78}$$

Then using the mean stochastic derivatives

$$\mathscr{D}\beta_*(X(t),t) = (\partial_t + \beta\cdot\nabla + \nu\Delta)\beta_*(x(t),t) \tag{79}$$

and

$$\mathscr{D}_*\beta(X(t),t) = (\partial_t + \beta_*\cdot\nabla - \nu\Delta)\beta(x(t),t) \tag{80}$$

in the evaluation of expression (78), we obtain

$$\partial_t v = a - (v\cdot\nabla)v + (u\cdot\nabla)u + \nu\Delta u \tag{81}$$

Equation (81) will be recognized readily as the gradient of the field equation (54).

To what extent is the classical stochastic interpretation successful in giving the correct quantum mechanical description of matter? We need not look beyond the case of the free particle to point out the shortcomings of the classical stochastic interpretation. From the decomposition of the wave field through the Ansatz (50), the components of the drift vector (73) are related to the wave function by

$$v = \frac{\hbar}{m}\operatorname{Im}\nabla\ln\psi \quad \text{and} \quad u = \frac{\hbar}{m}\operatorname{Re}\nabla\ln\psi \tag{82}$$

For the case of the free particle, $u = 0$ and

$$v = \hbar/m = \text{const} \tag{83}$$

The Fokker–Planck equation for the transition probability density of a free particle is

$$\partial_t p = -v\cdot\nabla p + \nu\Delta p \tag{84}$$

which has the well-known solution

$$p(x,t|0) = (4\pi\nu t)^{-d/2}\exp\{-|x - vt|^2/4\nu t\} \tag{85}$$

Clearly, expression (85) does not describe the periodic behavior of a free particle. Even an analytic continuation in time in which $t \to -it$ cannot render (85) periodic unless $v = 0$. In other words, only when the S field vanishes can an analytic continuation in time render the classical stochastic interpretation equivalent to quantum mechanics. The classical stochastic formulation cannot, in general, describe the nonclassical aspects of matter that result from the quantum phenomenon of interfering alternatives. Therefore, the time analytic continuation method can only achieve success in the classical stochastic description of bound states.

In order to derive the Schrödinger equation from the real field equations (77) and (81), it is necessary to construct the complex drift vector

(Santamato and Lavenda, 1981)

$$b(x,t) = v(x,t) - iu(x,t) \tag{86}$$

Consequently, we cannot demand that the Fokker–Planck equations for the hierarchy of probability densities be real. In particular, the probability density satisfies

$$\partial_t \rho = -\nabla\cdot(v - iu)\rho - i\frac{\hbar}{2m}\Delta\rho \tag{87}$$

If we now impose the condition that ρ be real, we obtain the continuity equation (58) and the formal Einstein condition of dynamic equilibrium (76) as consequences of this condition. Furthermore, it is logically inconsistent to invoke Newton's second law into the Einstein–Smoluchowski description (64) of Brownian motion (Lavenda, 1980). In the generalized stochastic interpretation, classical mechanics enters insofar as the external fields modify the probability distribution. The scalar potential enters the expression for the complex probability measure density through the generalized Hamilton–Jacobi equation. We now turn to a derivation of this equation by invoking average energy conservation in isolated systems and average energy balance in systems with an external vector potential (Santamato and Lavenda, 1981).

First, consider an isolated system. Independent of whether the system is isolated or not, the energy is defined classically

$$\varepsilon = -\partial_t \mathscr{S} \tag{88}$$

which is a function of the quantum mechanical diffusion process. Its mean stochastic derivative is

$$\mathscr{D}\varepsilon = \partial_t\varepsilon + b\cdot\nabla\varepsilon - i\frac{\hbar}{2m}\Delta\varepsilon \tag{89}$$

In contrast to the definition of energy (88), the definition of momentum will depend on whether the system is isolated or not. In isolated systems, it is given by

$$mb = \nabla\mathscr{S} \tag{90}$$

Introducing (90) into (89) and integrating in time yields

$$\partial_t\mathscr{S} + \frac{1}{2m}(\nabla\mathscr{S})^2 - i\frac{\hbar}{2m}\Delta\mathscr{S} = -V(x) \tag{91}$$

where the constant of integration, V, has dimensions of energy. It could be argued that via the correspondence principle, equation (91) must transform into the classical Hamilton–Jacobi equation in the classical limit as $\hbar \downarrow 0$. This would serve to identify V as the classical potential energy but it would not exclude the possibility that there would be other terms, proportional to $\hbar$, in the constant of integration. Rather, we must make appeal to Ehrenfest's theorem, which states that quantum processes, on the average, satisfy the laws of classical mechanics. This does not exclude terms involving $\hbar$ to appear in the average equations of motion, as we shall see in Section 5.3. however, it does imply that the mean stochastic derivative of the action $\mathscr{S}$ be equal to the classical Lagrangian, viz.,

$$\mathscr{D}\mathscr{S} = \partial_t \mathscr{S} + b \cdot \nabla \mathscr{S} - i\frac{\hbar}{2m}\Delta \mathscr{S} = \frac{1}{2m}(\nabla \mathscr{S})^2 - V(x) \tag{92}$$

Then on the strength of the generalized Hamilton–Jacobi equation (91), we identify V as the classical potential energy.

Second, consider the system in the presence of a vector potential $\mathscr{A}$. To be concrete, consider the physical space E^3 in which the rate of change of the particle energy in time is given by the expression

$$-\frac{e}{c}\int \mathbf{J} \cdot \partial_t \mathbf{A}\, d^3r$$

where $\mathbf{J}$ is the particle current density. Quantum mechanics reinterprets $\mathbf{J}$ as the quantum mechanical probability current density (Santamato and Lavenda, 1981)

$$\mathbf{J} = \frac{1}{m}\left(\nabla \mathscr{S} - \frac{e}{c}\mathbf{A}\right)\rho + i\frac{\hbar}{2m}\Delta\rho \tag{93}$$

The mean stochastic derivative of the energy is still given by expression (89) but the drift is now

$$\mathbf{b} = \frac{1}{m}\left(\nabla \mathscr{S} - \frac{e}{c}\mathbf{A}\right) \tag{94}$$

In order that there be a power balance, on the average, we must have the condition

$$\int \rho \mathscr{D}\varepsilon\, d^3r = -\frac{e}{c}\int \mathbf{J} \cdot \partial_t \mathbf{A}\, d^3r \tag{95}$$

Inserting the quantum mechanical probability current density (93) and integrating in time we obtain

$$\partial_t \mathcal{S} + \frac{1}{2m}\left(\nabla\mathcal{S} - \frac{e}{c}\mathbf{A}\right)^2 - i\frac{\hbar}{2m}\left(\Delta\mathcal{S} - \frac{e}{c}\nabla\cdot\mathbf{A}\right) = -\frac{e}{c}\Phi(\mathbf{r}) \qquad (96)$$

where several integrations by parts had to be performed prior to the final integration. Expression (96) is the condition for a stochastic balance of energy. Again invoking Ehrenfest's theorem, the constant of integration Φ is identified as the scalar potential.

The classical stochastic interpretation contends that applied fields act via the force to cause a change in the motion of the particle. Yet, a force, in the Newtonian sense, necessarily entails a reaction which is completely foreign to quantum theory. Rather, in the generalized stochastic formulation, the applied fields have been shown to modify the probability measure of the particle trajectories. And since the particle trajectories are unobservable (i.e., the probability measure is complex), the only observable effect of the applied fields is to modify the probability that the particle will be found in any given state (i.e., the probability distribution is real).

The realness of the probability density is a consequence of the fact that all applied fields are real. This explains why there is no vector potential appearing in the imaginary part of the drift vector (52). For if there were such a component, probability would not be conserved and this would entail major alterations in quantum theory. This is in support of the fact that quantum mechanics deals with real diffusion processes that are acted upon by real applied fields; albeit their associated probability measures are complex. The complexity of the probability measures for particle trajectories can be said to stand in the defense of quantum theory which claims that they are unobservable.

5. THE STOCHASTIC STRUCTURE OF QUANTUM MECHANICS

In this section, we illustrate the complete compatibility between the generalized stochastic approach, which uses the methods of classical probability theory and quantum theory by taking specific topics from quantum mechanics and showing how they are derived within the framework of the generalized stochastic formulation.

5.1. Commutation Relations. It is well known that the failure of conjugate variables to commute in quantum mechanics is related to the uncertainty relations which place a lower bound on the precision with which a simultaneous measurement can be performed. We shall attribute the uncertainty to the statistical correlations between the diffusion process and any dynamical variable which is a function of the diffusion process. In the generalized stochastic interpretation, every dynamic variable is a function of the quantum mechanical diffusion process and the statistical correlations between any function of the process and the increments in the process itself will lead to a lack of commutativity which prevents their simultaneous measurement with unlimited precision.

Consider the physical space E^3 and let $\mathbf{O}(\mathbf{r}(t), t)$ represent a dynamical variable which is a function of the quantum diffusion process $\mathbf{R}(t)$ and possibly time. The forward Itô stochastic integral of $\mathbf{O}$ is defined as (Itô, 1944)

$$\int_s^t \mathbf{O}(\mathbf{R}(\tau),\tau)\,d\mathbf{R}(\tau) = \lim_{\Delta\downarrow 0}\sum_{i=0}^{N-1}\mathbf{O}(\mathbf{r}(\tau_i),\tau_i)\left[\mathbf{r}(\tau_{i+1})-\mathbf{r}(\tau_i)\right] \tag{97}$$

where the interval $[s,t]$ has been divided into $N-1$ subdivisions, $\Delta = \max_i[\mathbf{r}(\tau_{i+1})-\mathbf{r}(\tau_i)]$, and the limit is understood to be in the mean (l.i.m.) (assuming the condition $E\{\int_s^t|\mathbf{O}(\mathbf{r}(\tau),\tau)|^2\,dt\}<\infty$ is fulfilled). Analogously, the backward stochastic integral is given by (Stratonovich, 1968)

$$\int_s^t d\mathbf{R}(\tau)\mathbf{O}(\mathbf{R}(\tau),\tau) = \lim_{\Delta\downarrow 0}\sum_{i=0}^{N-1}\left[\mathbf{r}(\tau_{i+1})-\mathbf{r}(\tau_i)\right]\mathbf{O}(\mathbf{r}(\tau_{i+1}),\tau_{i+1}) \tag{98}$$

Neither the forward (97) nor backward (98) stochastic integral obeys the formal rules of calculus. However, they can be used to construct the Fisk–Stratonovich stochastical integral (Stratonovich, 1966)

$$\int_s^t \mathbf{O}(\mathbf{R}(\tau),\tau)\circ d\mathbf{R}(\tau) = \begin{cases} \displaystyle\int_s^t \mathbf{O}(\mathbf{R}(\tau),\tau)\,d\mathbf{R}(\tau) - \frac{i\hbar}{2m}\int_s^t \nabla\cdot\mathbf{O}(\mathbf{R}(\tau),\tau)\,d\tau \\ \displaystyle\int_s^t d\mathbf{R}(\tau)\mathbf{O}(\mathbf{R}(\tau),\tau) + \frac{i\hbar}{2m}\int_s^t \nabla\cdot\mathbf{O}(\mathbf{R}(\tau),\tau)\,d\tau \end{cases} \tag{99}$$

which does obey the rules of ordinary calculus. The stochastic correction terms in (99) are related to the uncertainty in the final and initial states of transition, respectively (Lavenda and Santamato, 1979). In the midpoint definition of the stochastic integral [i.e., one-half the sum of (99)], the uncertainties in the end points of transition compensate one another and we obtain the Fisk–Stratonovich definition of the stochastic integral. On the other hand, if we take the difference between the two expressions in (99) we obtain

$$\int_s^t \{\mathbf{O}(\mathbf{R}(\tau),\tau)\,d\mathbf{R}(\tau)-d\mathbf{R}(\tau)\mathbf{O}(\mathbf{R}(\tau),\tau)\} = \frac{i\hbar}{m}\int_s^t \nabla\cdot\mathbf{O}(\mathbf{R}(\tau),\tau)\,d\tau \tag{100}$$

Expression (100) represents the uncertainties in the end points of transition; it is a prototype of the commutation relations in quantum mechanics, as we shall now show.

Let **O** represent the displacement operator $\hat{R}$. In differential form, it is easy to see that (100) is identical to the familiar commutation relation

$$[\hat{R},\hat{p}] = i\hbar \tag{101}$$

which is ordinarily obtained through the formal procedure of first quantization, $\hat{p} = -i\hbar\nabla$. It is important to bear in mind that this formal procedure has been superseded in expression (100) where the classical definition of momentum, $\mathbf{p} = m\,d\mathbf{R}/dt$ has been used.

Now let **O** denote the angular momentum operator $\hat{L}$. The usual commutation relations, given in the compact form (Schiff, 1968)

$$\hat{L}\times\hat{L} = i\hbar\hat{L} \tag{102}$$

is identical to (100) if we use the classical definition of angular momentum, $\mathbf{L} = \mathbf{R}\times\mathbf{p}$, instead of the first quantization expression $\hat{L} = -i\hbar\hat{R}\times\nabla$. In fact, any observable **O**, depending upon the coordinates, and having the operator representation $\hat{O}$, will satisfy the commutator relation

$$[\hat{O},\hat{p}] = i\hbar\nabla\cdot\mathbf{O}(\mathbf{r}) \tag{103}$$

Expression (103) brings out the intrinsic relationship between the operator formulation of quantum theory and the generalized stochastic approach in which the observable is considered as a function of the quantum mechanical diffusion process and (103) is essentially the integrand of the stochastic correction term in (99). The fact that **O** is a function of the quantum mechanical diffusion process leads to statistical correlations between the integrands in (100) and the increments in the process itself. It is precisely these statistical correlations, modeled as Brownian motion, which give rise to the quantum mechanical uncertainty relations.

5.2. The Virial Theorem. The quantum mechanical virial theorem

$$d_t\langle\{\mathbf{R}\cdot\mathbf{p}\}\rangle = \frac{1}{m}\langle\{\mathbf{p}^2\}\rangle - \int\psi^*\mathbf{R}\cdot\nabla V\psi\,d^3r = \frac{1}{i\hbar}\langle\{[\hat{R}\cdot\hat{p},\hat{H}]\}\rangle = 0 \tag{104}$$

where $\hat{H}$ is the Hamiltonian, can be derived in a completely classical way using the methods of classical probability theory. To this end, we employ

$$\mathscr{D}O(\mathbf{R}(t),t) = \lim_{\tau\downarrow 0}\tau^{-1}E\{O(\mathbf{R}(t+\tau),t+\tau)-O(\mathbf{R}(t),t)|\mathsf{P}_t\} \tag{105}$$

which is the mean stochastic derivative of the smooth function $O(\mathbf{r}(t),t)$ along a sample path of the stochastic process $\mathbf{R}(t)$. The mean forward derivative of the process itself [cf. (65)] is a special case of (105). With the aid of the Itô chain rule of stochastic calculus, we have

$$\mathscr{D}O(\mathbf{R}(t),t) = (\partial_t + \mathbf{b}\cdot\nabla - iD:\nabla\nabla)O(\mathbf{r}(t),t) \tag{106}$$

We now make the observation that the mathematical expectation is equivalent to the quantum mechanical rule for calculating averages, viz.,

$$E\{O(\mathbf{R}(t),t)\} = \int O(\mathbf{r},t)\rho(\mathbf{r},t)\,d^3r = \int\psi^*(\mathbf{r},t)\hat{O}\psi(\mathbf{r},t)\,d^3r \tag{107}$$

which can easily be demonstrated by differentiating (107) in time. Classically, we have

$$d_tE\{O(\mathbf{R}(t),t)\} = E\{\mathscr{D}O(\mathbf{R}(t),t)\} \tag{108}$$

while quantum mechanically we get

$$d_t\langle\hat{O}\rangle = \langle\partial_t\hat{O}\rangle + \frac{1}{2m}\langle[\hat{O},\hat{H}]\rangle \tag{109}$$

which will be recognized as the Heisenberg equation of motion where $\langle\cdot\rangle = \int\psi^*\cdot\psi\,d^3r$. Now

$$d_t\int O\rho\,d^3r = \int\rho[\partial_t + \mathbf{b}\cdot\nabla - iD:\nabla\nabla]O(\mathbf{r},t)\,d^3r \equiv \int\rho(\partial_t + \mathscr{G})O\,d^3r \tag{110}$$

and comparing (109) and (110) we obtain

$$\langle[\hat{O},\hat{H}]\rangle = i\hbar E\{\mathscr{G}O(\mathbf{R}(t),t)\} \tag{111}$$

where $\mathscr{G}$ is the infinitesimal generator of the quantum mechanical diffusion process, defined by (110). In deriving (110), we have used the complex Fokker–Planck equation (87) for the probability density and have performed several integrations by parts.

Expression (111) is our fundamental result: it shows that the average temporal evolution of quantum mechanical observables is determined by the infinitesimal generator of the diffusion process in which the "systematic" drift vector is "guided" by the wave function according to the relation

$$\mathbf{b} = -\frac{i\hbar}{m} \nabla \ln \psi \tag{112}$$

in the absence of an external vector potential. If we now set $O = \mathbf{R} \cdot \mathbf{p}$ in expression (108) we get

$$d_t \langle \mathbf{R} \cdot \mathbf{p} \rangle = mE\{\mathscr{D}(\mathbf{b} \cdot \mathbf{R})\} \tag{113}$$

which with the aid of the gradient of the generalized Hamilton–Jacobi equation

$$\partial_t \mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \frac{i\hbar}{2m} \Delta \mathbf{b} = -\frac{1}{m} \nabla V \tag{114}$$

becomes

$$d_t \langle \mathbf{R} \cdot \mathbf{p} \rangle = -E\{\nabla V \cdot \mathbf{R}\} + mE\{\mathbf{b}^2\} \tag{115}$$

In a stationary state, (115) vanishes giving

$$mE\{\mathbf{b}^2\} = E\{\mathbf{R} \cdot \nabla V\} \tag{116}$$

which is precisely the virial theorem since the left-hand side is twice the average kinetic energy.

5.3. Ehrenfest's Theorem. Quantum theory considers the time-rate-of-change of operator averages corresponding to physical observables. Since the time-rate-of-change of the operator averages obey the classical equations of motion (i.e., Ehrenfest's theorem), the same should be true of the mathematical expectations of observables which are functions of the quantum mechanical diffusion process. Although in the classical limit the width of the wave packets become negligible and the average quantum mechanical equations of motion transform into the equations of motion of classical mechanics, quantum theory makes a more stringent demand by requiring the average quantum mechanical equations of motion to always coincide with the classical equations of motion. This is the essence of Ehrenfest's theorem, which we shall now derive in the context of the generalized stochastic formulation (Santamato and Lavenda, 1981).

Again consider the physical space E^3 in which a charged particle interacts with an external electromagnetic field $\mathbf{A}(\mathbf{r}, t)$. The classical Hamiltonian is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi \tag{117}$$

The averaged quantum mechanical equations of motion are

$$d_t\langle\mathbf{R}\rangle = \frac{1}{m}\langle\mathbf{p} - \frac{e}{c}\mathbf{A}\rangle \tag{118}$$

and

$$md_t\langle\dot{\mathbf{R}}\rangle = e\langle\mathbf{E}\rangle + \frac{1}{2}\frac{e}{c}\langle\dot{\mathbf{R}}\times\mathbf{B} - \mathbf{B}\times\dot{\mathbf{R}}\rangle \tag{119}$$

where $\mathbf{e}$ and $\mathbf{B}$ are the electromagnetic field strengths that satisfy the field equations (60) and $\dot{\mathbf{R}} \equiv d_t\mathbf{R}$ is the velocity of the particle. The striking nonclassical appearance of the last term in equation (119) has the explanation that since $\dot{\mathbf{R}}$ does not commute with $\mathbf{B}$, neither $\dot{\mathbf{R}}\times\mathbf{B}$ nor $-\mathbf{B}\times\dot{\mathbf{R}}$ is Hermitian but rather their sum is Hermitian and corresponds to the classical term $\dot{\mathbf{R}}\times\mathbf{B}$ (Schiff, 1968, p. 179).

Since the generalized stochastic interpretation treats "processes" rather than "operators," it would appear, at first sight, that it would be unable to distinguish between the quantum and classical equations of motion. However, we now show this not to be the case.

Consider the equation of motion (118). The mean forward stochastic velocity is

$$d_tE\{\mathbf{R}(t)\} = E\{\mathbf{b}(\mathbf{R}(t), t)\} = \frac{1}{m}E\left\{\nabla\mathscr{S} - \frac{e}{c}\mathbf{A}\right\} \tag{120}$$

where we have used expression (15) with $\mathscr{A} = -(e/c)\mathbf{A}$. As for the second equation of motion (119), the mathematical expectation of the mean stochastic drift vector is

$$E\{\mathscr{D}\mathbf{b}(\mathbf{R}(t), t)\} = E\left\{\partial_t\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \frac{i\hbar}{2m}\Delta\mathbf{b}\right\} \tag{121}$$

is to be evaluated with the aid of the gradient of the generalized Hamilton–Jacobi equation (23), viz.,

$$\partial_t\mathbf{b} + \frac{e}{mc}\partial_t\mathbf{A} + \tfrac{1}{2}\nabla b^2 - \frac{i\hbar}{2m}\nabla(\nabla\cdot\mathbf{b}) = -\frac{e}{m}\nabla\Phi \tag{122}$$

In equation (122), we have identified $e\Phi$ as the scalar potential, V. On the basis of the field equations (60) and the vector identities

$$\tfrac{1}{2}\nabla\mathbf{b}^2=(\mathbf{b}\cdot\nabla)\mathbf{b}+\mathbf{b}\times(\nabla\times\mathbf{b})$$

and

$$\nabla(\nabla\cdot\mathbf{b})=\Delta\mathbf{b}+\nabla\times(\nabla\times\mathbf{b})$$

Equation (122) can be written as

$$\partial_t\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{b}-\frac{i\hbar}{2m}\Delta\mathbf{b}=\frac{e}{m}\mathbf{E}-\mathbf{b}\times(\nabla\times\mathbf{b})+\frac{i\hbar}{2m}\nabla\times(\nabla\times\mathbf{b}) \tag{123}$$

Now inserting equation (123) into equation (121) results in

$$mE\{\mathscr{D}\mathbf{b}(\mathbf{R}(t),t)\}=E\left\{\mathbf{F}-\frac{i\hbar e}{2mc}\nabla\times\mathbf{B}\right\} \tag{124}$$

where **F** is the Lorentz force, (62), and the identity

$$\nabla\times\mathbf{b}=-\frac{e}{mc}\nabla\times\mathbf{A}=-\frac{e}{mc}\mathbf{B}$$

has been used.

At first glance, the averaged equation of motion (124) would appear to contradict Ehrenfest's theorem on account of the term involving Planck's constant. Surprisingly enough, it is this term which renders the operator $-(i\hbar\nabla+(e/c)\mathbf{A})$ Hermitian. This is tantamount to our condition that equation (124) be real (Santamato and Lavenda, 1981). In order to demonstrate the equivalence between the averaged equations of motion, (119) and (124), it suffices to introduce the operator $-(1/m)[i\hbar\nabla+(e/c)\mathbf{A}]$ for $\dot{\mathbf{R}}$ on the right-hand side of equation (119). We thus obtain

$$md_t\langle\dot{\mathbf{R}}\rangle=e\langle\mathbf{E}\rangle-\frac{e}{mc}\int\left\{\psi^*\left[\left(i\hbar\nabla\psi+\frac{e}{c}\mathbf{A}\psi\right)\times\mathbf{B}-\tfrac{1}{2}i\hbar(\nabla\times\mathbf{B})\psi\right]d^3r\right\} \tag{125}$$

Since the sum of the first two terms in equation (125) is the Lorentz force **F**, the equivalence between equations (119) and (124) has been established.

5.4. The Quasiclassical Approximation. We now establish the equivalence between the conventional WKB approximation for the wave function and the same approximation for the complex transition probability density in the near classical limit as $\hbar \downarrow 0$. By virtue of the logarithmic transformation (24), the complex transition probability (32) can be cast into the form

$$p(x,t|y,s)=\exp\left\{\frac{i}{\hbar}\left[\mathscr{S}(x,t)-\mathscr{S}(y,s)+\Omega^*(x,t|y,s)\right]\right\} \tag{126}$$

where we have set

$$\Omega(x,t|y,s)=i\hbar\ln K(x,t|y,s) \tag{127}$$

The complex conjugate of (127) accounts for the statistical correlations between the state of the system and the measuring apparatus since Ω^* is uniquely defined by the applied external fields. The joint action Ω is seen to satisfy the pair of generalized Hamilton–Jacobi equations

$$\partial_t\Omega^*=\frac{1}{2m}(\nabla\Omega^*)^2-\frac{i\hbar}{2m}\Delta\Omega^*+V(x) \tag{128}$$

and

$$-\partial_s\Omega^*=\frac{1}{2m}(\nabla'\Omega^*)^2-\frac{i\hbar}{2m}\Delta'\Omega^*+V(y) \tag{129}$$

where the prime denotes differentiation with respect to the initial coordinates.

We look for a solution to equations (128) and (129) in the form of an asymptotic expansion

$$\Omega^*=\Omega_0^*+\left(\frac{\hbar}{i}\right)\Omega_1^*-\tfrac{1}{2}\hbar^2\Omega_2^*-\cdots \tag{130}$$

which when substituted back into equations (128) and (129) generates an infinite number of coupled equations by setting terms of the same power in Planck's constant equal to zero. To lowest order we have

$$-\partial_t\Omega_0^*+\frac{1}{2m}(\nabla\Omega_0^*)^2+V(x)=0 \tag{131}$$

$$\partial_s\Omega_0^*+\frac{1}{2m}(\nabla'\Omega_0^*)^2+V(y)=0 \tag{132}$$

which are none other than the classical Hamilton–Jacobi equations for the classical action S_c. Hence

$$\Omega_0^*(x,t|y,s)=\Omega_0(x,t|y,s)=S_c(x,t|y,s) \tag{133}$$

and the classical velocities of the end points at the transition are

$$v_c(x,t) = -\frac{1}{m}\nabla S_c, \qquad v'_c = \frac{1}{m}\nabla' S_c \tag{134}$$

The first-order correction to the classical result (133) is obtained by solving

$$\partial_t \Omega_1^* = \frac{1}{m}\nabla\Omega_0^*\nabla\Omega_1^* + \frac{1}{2m}\Delta\Omega_0^* \tag{135}$$

$$-\partial_s \Omega_1^* = \frac{1}{m}\nabla'\Omega_0^*\nabla'\Omega_1^* + \frac{1}{2m}\Delta'\Omega_0^* \tag{136}$$

Upon introducing the classical velocities (134) we get

$$\partial_t \Omega_1^* = -v_c\cdot\nabla\Omega_1^* - \tfrac{1}{2}\nabla\cdot v_c \tag{137}$$

$$\partial_s \Omega_1^* = -v'_c\cdot\nabla'\Omega_1^* - \tfrac{1}{2}\nabla'\cdot v'_c \tag{138}$$

Equations (137) and (138) manifest a symmetry in past and future, viz.,

$$\Omega_1^*(x,t|y,s) = \Omega_1^*(y,s|x,t) \tag{139}$$

which is a consequence of the reversible nature of the quantum diffusion process.

Equations (137) and (138) can be cast into a physically more intuitive form by performing the logarithmic transformation

$$2\Omega_1^* = \ln\sigma \tag{140}$$

which converts them into the conservation law

$$\partial_t\sigma + \nabla\cdot v_c\sigma = 0 \tag{141}$$

The solution to the continuity equation (141) can be found by considering the forward classical Hamilton–Jacobi equation (131) (Santamato and Lavenda, 1980). In component form it reads

$$\partial_t S_c + \frac{1}{2m}(\partial_i S_c)(\partial_i S_c) + V(x) = 0 \tag{142}$$

Differentiating with respect to the initial coordinate and using the definition of the backward, classical velocity (134), we obtain

$$m\left(\partial_t v'_{ci} + v_{cj}\partial_j v'_{ci}\right) = 0 \tag{143}$$

Now differentiating with respect to the final coordinates and setting $Q = [-\partial_i \partial'_j S_c]$ and multiplying through by Q^{-1} results in

$$Q^{-1} : (\partial_t Q + v_{ci} \partial_i Q) + \partial_i v_{ci} = 0 \tag{144}$$

A comparison of equations (141) and (144) leads to the conclusion that

$$\sigma = |-\partial_i \partial'_j S_c| \tag{145}$$

which is the well-known Van Vleck determinant. Therefore, in the quasiclassical approximation, the complex transition probability density is

$$p(x, t|y, s) \sim \frac{\psi(x, t)}{\psi(y, s)} \{|-\partial_i \partial'_j S_c|\}^{1/2} \cdot \exp\left\{-\frac{i}{\hbar} S_c(x, t|y, s)\right\} \tag{146}$$

Then comparing (146) with expression (32) we find that the Feynman kernel is

$$K(x, t|y, s) \sim \{|-\partial_i \partial'_j S_c|\}^{1/2} \exp\left\{-\left[\frac{i}{\hbar} S_c(x, t|y, s)\right]\right\} \tag{147}$$

which demonstrates the complete equivalence between conventional quantum theory and the generalized stochastic interpretation in the quasiclassical limit (Santamato and Lavenda, 1980).

5.5. Quantum Mechanical Correlation Functions. In Section 5.2, we have shown that mathematical expectations of functions of the quantum mechanical diffusion process correspond to the quantum mechanical averages of operators in the traditional formulation. Even more can be said when we consider the set of multitime moments in the generalized stochastic formulation, and in particular the two-time moments or correlation functions. It will now be appreciated that the generalized stochastic formulation offers a straightforward generalization of the classical correlation functions to quantum theory. However, it should be borne in mind that the resulting expression will differ from those of classical probability theory since we can no longer impose the conditions of positivity or realness on the moments. This is attributed to the presence of a complex transition probability density in calculating the averages, and the form of the multitime moments will appear as another manifestation of the autoselectiveness of the generalized stochastic approach. Only physical observables will turn out to be real, averaged quantities.

The two time correlation function of any two functions, f and g, of the quantum mechanical diffusion process is defined by the classical expression

$$E\{f(X(t))g(X(s))\} \equiv \iint dx\,dy\,f(x)g(y)p(x,t|y,s)\rho(y,s) \quad (148)$$

The correlation function so defined possesses the following properties:

(i) Compatibility

$$E\{f(X(t))1(X(s))\} = E\{f(X(t))\}$$

where $1(x)$ is the unit function,

(ii) Linearity

$$E\{f(X(t))[\alpha g(X(s))+\beta h(X(s))]\} = \alpha E\{f(X(t))g(X(s))\} + \beta E\{f(X(t))g(X(s))\}$$

(iii) Symmetry

$$E\{f(X(t))g(X(s))\} = E\{g(X(s))f(X(t))\}$$

but not positivity or reality. The symmetric operator ordering rule is equivalent to taking the real part of the correlation function, viz.,

$$\tfrac{1}{2}\langle \hat{f}\hat{g} + \hat{g}\hat{f}\rangle = \operatorname{Re} E\{f(X(t))g(X(s))\} \quad (149)$$

The definition of quantum mechanical correlation functions provides a good testing ground for the comparison of the classical and generalized stochastic approaches.

On the basis of the harmonic oscillator, Guerra and Ruggiero (1973) have argued that the process corresponding to the ground state in the classical stochastic approach is equivalent to a Brownian motion with imaginary time. We now show that the time analytic continuation method gives the correct quantum mechanical result only in the case that the real current velocity v vanishes (cf. Section 4). Furthermore, it is immaterial whether one considers the ground state or an excited state. We shall attribute this to the property of bound states in which the wave properties of matter do not come into play.

According to the classical stochastic interpretation, the real drift is given by expression (73), which on account of the formal condition of dynamical equilibrium, (76), can be written as

$$\beta_0(x,t) = v(x,t) + \frac{\hbar}{2m}\partial_x \ln \rho_0(x,t) \quad (150)$$

in E^1 where the subscript denotes the ground state. In the case of the harmonic oscillator, the ground state wave function is

$$\phi_0(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2-i\omega t\right) \tag{151}$$

so that the drift associated with this state is

$$\beta_0(x)=-\omega x \tag{152}$$

The diffusion process associated with the ground state of the harmonic oscillator is described by the stochastic differential equation

$$dX(t)=-\omega X(t)\,dt+\left(\frac{\hbar}{m}\right)^{1/2}dW(t) \tag{153}$$

where W is a real probability measured (standard) Wiener process.

Since equation (153) is simply the Ornstein–Uhlenbeck process on configuration space, we know that the transition probability density is

$$p_0(x,\tau|y)=\left\{\frac{m\omega}{\pi\hbar[1-\exp(-2\omega\tau)]}\right\}^{1/2}\exp\left\{-\frac{m\omega}{\hbar}\frac{(x-ye^{-\omega\tau})^2}{[1-\exp(-2\omega\tau)]}\right\} \tag{154}$$

Then invoking time analytic continuation, $\tau\to -i\tau$, it is easy to see that (154) propagates the ground state probability density,

$$\rho_0(y)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega}{\hbar}y^2\right) \tag{155}$$

according to (9), which in this case is

$$\rho_0(x,\tau)=\int p_0(x,-i\tau|y)\rho_0(y)\,dy \tag{156}$$

We may immediately generalize the time analytic continuation method to excited states, provided $v=0$. The drift corresponding to the first excited state is

$$\beta_1(x)=\frac{\hbar}{m}\partial_x\ln\phi_1(x,t) \tag{157}$$

where ϕ_1 is the wave function of the first excited state

$$\phi_1(x,t) = \frac{2m\omega}{\hbar} x\phi_0(x,t)\exp(-i\omega t) \tag{158}$$

And consequently, the Fokker–Planck equation for the real transition probability density of the first excited state of the harmonic oscillator is

$$\partial_t p_1 = -\partial_x\left\{\left[\frac{\hbar}{m}\frac{1}{x} - \omega x\right]p_1 - \frac{\hbar}{2m}\partial_x p_1\right\} \tag{159}$$

which is simply the Rayleigh process (Stratonovich, 1963). The solution to equation (159) may be expressed in the form [cf. equation (32)]

$$p_1(x,\tau|y) = \frac{\phi_1(x,t)}{\phi_1(y,0)}K(x,\tau|y) \tag{160}$$

where K is the real kernel

$$K(x,\tau|y) = \left(\frac{m\omega}{2\pi\hbar\sinh\omega\tau}\right)^{1/2} \cdot\exp\left\{-\frac{m\omega}{2\hbar\sinh\omega\tau}\left[(x^2+y^2)\cosh\omega\tau - 2xy\right]\right\} \tag{161}$$

Here again, a time analytic continuation of the kernel (161) [but not the transition probability density (160)!] gives the correct quantum mechanical result. Upon performing the time analytic continuation $\tau \rightarrow -i\tau$, (161) transforms into the complex conjugate of the Feynman kernel and (160) then propagates the probability density of the first excited state of the harmonic oscillator according to (Santamato and Lavenda, 1981)

$$\phi_1(x,\tau) = \int p_1(x,-i\tau|y)\rho_1(y)\,dy \tag{162}$$

where

$$p_1(x,-i\tau|y) = \frac{\phi_1(x,\tau)}{\phi_1(y,0)}K(x,-i\tau|y) = \frac{\phi_1(x,\tau)}{\phi_1(y,0)}K^*(x,\tau|y)$$

The reason why analytic continuation works for the kernel and not for the transition probability density is that the first factor in (160) describes the interference of the state with itself at two different times.

To understand why the time analytic continuation procedure does not, in general, give the correct quantum mechanical results, it suffices to consider the initially displaced harmonic oscillator. At time $t=0$, the harmonic oscillator is in the ground state

$$\phi_0(x_0)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}(x_0-a)^2\right] \tag{163}$$

which is centered about the position $x=a$. At a later time t, the state of the system is

$$\begin{aligned}\phi_0(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\Big[&-\frac{1}{2}\frac{m\omega}{\hbar}(x-a\cos\omega t)^2\\ &-\frac{im\omega}{\hbar}(ax\sin\omega t-\tfrac{1}{4}a^2\sin 2\omega t-\tfrac{1}{2}i\omega t\Big]\end{aligned} \tag{164}$$

which can either be obtained by solving Schrödinger's equation or applying time analytic continuation to the kernel (161) and then using the propagator relation

$$\phi_0(x,t)=\int K(x,it|y,is)\phi_0(y,s)\,dy \tag{165}$$

where

$$\phi_0(y,s)=\int K(y,is|x_0)\phi_0(x_0)\,dx_0 \tag{166}$$

The drift corresponding to the ground state is now given by the expression

$$\beta_0(x,t)=-\omega[x-a(\cos\omega t-\sin\omega t)] \tag{167}$$

so that the real Fokker-Planck equation is

$$\partial_t p_0=\omega\partial_x[x-a(\cos\omega t-\sin\omega t)]\,p_0+\frac{\hbar}{2m}\partial_x^2 p_0 \tag{168}$$

The solution to equation (168) is easily found to be

$$\begin{aligned}p_0(x,t|y,s)=&\left\{\frac{m\omega}{\pi\hbar[1-\exp(-2\omega(t-s))]}\right\}^{1/2}\\ &\times\exp\left\{-\frac{m\omega\left[x-ye^{-\omega(t-s)}-a(\cos\omega t-e^{-\omega(t-s)}\cos\omega s)\right]^2}{\hbar[1-\exp(-2\omega(t-s))]}\right\}\end{aligned} \tag{169}$$

In contrast to the transition probability density (154) for $a \equiv 0$, the time analytic continuation of (169) does not lead to the correct quantum mechanical result (Grabert et al., 1979). This is clearly seen by calculating the correlation function, over the ground state. The result is

$$E\{X(t)X(s)\} = \iint dx\, dy\, xy p_0(x,t|y,s)\rho_0(y,s)$$
$$= \frac{\hbar}{2m\omega}\exp[-\omega(t-s)] + a^2\cos\omega t\cos\omega s \qquad (170)$$

It is now apparent that a time analytic continuation of (170) does not describe the reversible behavior of the quantum mechanical harmonic oscillator (Grabert et al., 1979).

The reason why the analytic continuation method for real Markov diffusion processes does not reproduce the quantum mechanical results is due to the fact that the current velocity, $v = -a\sin\omega t$, does not vanish unless a does. Hence we can expect the nonclassical aspects of matter to manifest themselves. It is also to be noted that the real diffusion process is not stationary in the wide sense. If, on the other hand, we apply the generalized stochastic approach,

$$b_0(x,t) = i\omega[x - a\exp(-i\omega t)] \qquad (171)$$

is identified as the complex drift for the ground state of the initially displaced harmonic oscillator. The corresponding complex Fokker–Planck equation for the ground state is

$$\partial_t p_0 = -i\omega\partial_x[x - a\exp(-i\omega t)]\, p_0 - \frac{i\hbar}{2m}\partial_x^2 p_0 \qquad (172)$$

which is to be solved subject to the initial condition (34). The solution to the complex Fokker–Planck equation (172) is

$$p_0(x,t|y,s) = \left\{\frac{m\omega}{\hbar\pi[1-\exp(2i\omega(t-s))]}\right\}^{1/2}$$
$$\times\exp\left\{-\frac{m\omega[x - ye^{i\omega(t-s)} - a(\cos\omega t - e^{i\omega(t-s)}\cos\omega s)]^2}{\hbar[1-\exp(2i\omega(t-s))]}\right\}$$
$$(173)$$

Instead of the correlation function expression (170) we now have

$$E\{X(t)X(s)\} = \frac{\hbar}{2m\omega}\exp[i\omega(t-s)] + a^2\cos\omega t\cos\omega s \qquad (174)$$

For $t = s$, both correlation functions, (170) and (174), give the correct second moment

$$E\left\{ X(t)^2 \right\} = \frac{\hbar}{2m\omega} + a^2\cos^2\omega t \tag{175}$$

of the harmonic oscillator. However, for $t > s$, it is only (174) that correctly describes the periodic statistical correlations of the quantum mechanical oscillator. Finally, the real part of the correlation function expression (174) coincides with the expression obtained by applying the symmetrization rule of quantum mechanics (Shewell, 1957) [cf. expression (149)].

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